nuclear size seen. The absolute diamagnetic shielding has been seen, perhaps for the first time, and verified to about 20%. Solid-state and chemical effects have apparently been seen, but their meaning in this experiment is not clear. Until these latter effects have been elucidated, either theoretically or by further measurements, they will serve to obscure a closer study of nuclear effects. It may be that the solid state and chemical shifts will become a subject for investigation in their own right, the muon serving as a tool for probing them.

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Asymptotic Invariants in Gravitational Radiation Fields

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Asymptotic integral invariants are constructed for the electromagnetic and gravitational fields. The integrals are taken over closed two-dimensional surfaces embedded in a null hypersurface. In the absence of incoming radiation, the asymptotic behavior of the electromagnetic field F^{ab} and the Riemann tensor R^{abcd} is such that the integrals formed with these quantities are independent of the particular space-like surface of integration, as long as it lies in the same null hypersurface. Therefore, the integrals are related to the multipole structure of the charge distribution and the matter distribution, respectively. This relationship is shown explicitly for the electromagnetic field and for the linearized gravitational field. It follows that energy radiation as determined by the Einstein pseudotensor depends on the existence of a type III asymptotic behavior of the Riemann tensor. Finally, the asymptotic conditions are formulated under which the superpotential $U_m{}^{ns}$ will also lead to asymptotically invariant integrals. It is pointed out that the linearized gravitational field with retarded potentials satisfies these conditions as do the asymptotic solutions for the Einstein field equations $R_{ab}=0$, which have been constructed by Bondi and Newman. The significance of this result for the interpretation of the Bondi metric is discussed.

1. INTRODUCTION

THE classification of the Riemann tensor for an Einstein space constructed by Petrov[1] was given its preliminary physical interpretation by Pirani[2] who identified certain of the special Petrov classes with the existence of gravitational radiation. In the following years, a distinction was drawn between the pure gravitational radiation field, corresponding to plane waves in the electromagnetic field, and an asymptotic gravitational field which may result from a matter distribution.[3–5] Thus, the existence of a pure gravitational radiation field leads to one of the algebraically special Petrov classes, in accord with Pirani, whereas a field with explicit sources belongs to the most general Petrov class and may become algebraically special at large distances. The purpose of this paper is to describe an additional tool, namely, asymptotically invariant integrals, for investigating the physical significance of vacuum gravitational fields, $G_{ab}=0$, particularly those containing radiation.

There have been two different approaches to the study of the asymptotic gravitational field, one looking at the properties of the Riemann tensor and the other examining the asymptotic behavior of the metric tensor. The Petrov classification has been shown to be related to the existence of preferred null directions at each point of space-time.[5–7] In fact, when the Riemann tensor is algebraically special, there always exists a congruence of shear-free null geodesics.[8–10] Sachs[5] used the properties of null geodesic congruences to discuss the propagation of the Riemann tensor along the null rays. From the explicit distance dependence in the algebraically

* The major portion of this research was performed while the author was on leave at Kings College, University of London, as a National Science Foundation Senior Post-Doctoral Fellow.

[1] A. Z. Petrov, Uch. Zap. Kazanskii Gos. Univ. **114**, 55 (1954).

[2] F. A. E. Pirani, Phys. Rev. **105**, 1089 (1957).

[3] C. W. Misner, Proceedings Chapel Hill Conference on the Role of Gravitation in Physics, 1957 (unpublished).

[4] A. Trautman, "Lectures on General Relativity," King's College, London, 1958 (unpublished).

[5] R. K. Sachs, Proc. Roy. Soc. (London) **264**, 309 (1961).

[6] R. Debever, Bull. Soc. Belge Math. **10**, 112 (1958).

[7] R. Penrose, Ann. Phys. **10**, 171 (1960).

[8] P. Jordan, J. Ehlers, and R. K. Sachs, Akad. Wiss. Lit. Mainz, Abhandl. Math. Nat. Kl. No. 1 (1961). Referred to in the text as (JES 1961).

[9] J. N. Goldberg and R. K. Sachs, Acta Phys. Polon. **22**, Suppl., 13 (1962).

[10] E. Newman and R. Penrose, J. Math. Phys. **3**, 566 (1962).

special cases, he was able to formulate an asymptotic condition on the Riemann tensor which implies the absence of incoming (or outgoing) radiation. This condition has been made more explicit by Newman and Penrose[10] who have proven that the Riemann tensor will fall off as $1/r$ if there exists a congruence of geodesic null rays whose tangent vectors are asymptotically one of the preferred null directions mentioned above. Bondi,[11] on the other hand, has studied the asymptotic behavior of solutions of the gravitational field equations. His method is based on the assumption that in the absence of incoming radiation, and for an appropriately chosen radial coordinate r, the metric tensor has an expansion in powers of $(1/r)$. In the axially symmetric case, to which he restricts his attention, Bondi is able to show that a loss of mass (or energy) necessarily results when a system which is initially at rest carries out a time-dependent motion and then returns to rest. Similar results have also been obtained by Newman and Unti.[12]

Since both the Sachs condition and the Bondi metric make statements only about the asymptotic field, neither can describe the matter distribution directly. Therefore, it is of some interest to see whether one can make invariant statements about the matter distribution, particularly in the presence of gravitational radiation. The method used here to construct asymptotically invariant quantities was suggested by the use of surface integrals in studying the equations of motion for the sources of the gravitational field.[13,14]

Consider a two-dimensional closed surface S which lies outside the localized matter distribution and form[15]

$$\mathcal{G}[w^m] \overset{\text{def}}{=} -2\oint_S w^m(-g)^{1/2}G_m{}^\alpha n_\alpha dS\,, \qquad (1.1)$$

where w^m is an arbitrary vector function and dS is the intrinsic element of surface area with the three-vector n_α as the outward normal. When the field equations $G_m{}^n=0$ are satisfied on S, clearly $\mathcal{G}=0$. However, in conjunction with an approximation method these integrals may be evaluated before all the field equations have been solved. When the w^m are the generators for an infinitesimal Lorentz transformation, one obtains the equations of motion.[16] In the following, on the other hand, it is always understood that the field equations are satisfied and, hence, $\mathcal{G}=0$.

Thus, in the above form Eq. (1.1) is trivial and can give no information. However, the field equations can be written in terms of a superpotential and the Einstein pseudotensor as follows[17,18]:

$$-2(-g)^{1/2}G_m{}^n = U_m{}^{[ns]}{}_{,s} - t_m{}^n = 0\,, \qquad (1.2)$$

with

$$U_m{}^{[ns]} = [1/(-g)^{1/2}]g_{mk}[(-g)(g^{kn}g^{rs}-g^{ks}g^{rn})]_{,r}\,, \qquad (1.3a)$$

$$t_m{}^n = (-g)^{1/2}\Bigg[g^{ns}\left(\begin{Bmatrix} a \\ ab\end{Bmatrix}\begin{Bmatrix} b \\ ms\end{Bmatrix} - \begin{Bmatrix} a \\ as\end{Bmatrix}\begin{Bmatrix} b \\ mb\end{Bmatrix}\right) + g^{rs}\left(\begin{Bmatrix} n \\ sr\end{Bmatrix}\begin{Bmatrix} b \\ mb\end{Bmatrix} + \begin{Bmatrix} b \\ sb\end{Bmatrix}\begin{Bmatrix} n \\ mr\end{Bmatrix} - 2\begin{Bmatrix} n \\ ar\end{Bmatrix}\begin{Bmatrix} a \\ ms\end{Bmatrix}\right) - \delta_m{}^n g^{rs}\left(\begin{Bmatrix} a \\ rs\end{Bmatrix}\begin{Bmatrix} b \\ ab\end{Bmatrix} - \begin{Bmatrix} b \\ ra\end{Bmatrix}\begin{Bmatrix} a \\ bs\end{Bmatrix}\right)\Bigg]. \qquad (1.3b)$$

The existence of the skew superpotential is equivalent to the contracted Bianchi identities which are satisfied by the field equations. With the substitution of Eq. (1.2) for the field equations, (1.1) becomes[19,20]

$$\frac{d}{dx^0}\oint w^m U_m{}^{[0\alpha]}n_\alpha dS = -\oint [w^m{}_{,s}U_m{}^{[s\alpha]} + w^m t_m{}^n]n_\alpha dS\,. \qquad (1.4)$$

This equation is to be understood as a continuity equation: *The rate of change of a certain quantity within the surface is determined by a corresponding flux through the surface.* In general, the quantities

$$U'[w^m] \overset{\text{def}}{=} \oint w^m U_m{}^{[\alpha 0]}n_\alpha dS \qquad (1.5)$$

will depend on the gravitational field as well as on the matter within S. In order to describe properties of the matter distribution alone, it is necessary to choose w^m so that U' is independent of the particular surface of integration, as long as the matter is wholly contained within S. The condition for this surface independence is

$$(w^m U_m{}^{[\alpha 0]})_{,\alpha} = 0\,. \qquad (1.6)$$

This condition may be satisfied if a Killing vector exists.[16] However, in general, Killing vectors do not exist and in the absence of additional assumptions it is not clear what restrictions are implied by Eq.(1.6).From the work of Sachs and Bondi mentioned above, one has information about the behavior of the asymptotic gravi-

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[11] H. Bondi, M. G. J. van der Burg, and A. W. K. Metzner, Proc. Roy. Soc. (London) **A269**, 21 (1962).

[12] E. T. Newman and T. W. J. Unti, J. Math. Phys. **3**, 891 (1962).

[13] A. Einstein, L. Infeld, and B. Hoffman, Ann. Math. **39**, 66 (1938).

[14] J. N. Goldberg, in *Gravitation*, edited by J. Witten (John Wiley & Sons, Inc., New York, 1962).

[15] Latin indices have the range 0, 1, 2, 3, whereas Greek indices have the range 1, 2, 3. The signature of the metric is −2.

[16] J. N. Goldberg, in *Recent Developments in General Relativity* (Państwowe Wydawnictwo Naukowe, PWN—Polish Scientific Publishers, Warsaw and Pergamon Press Inc., London, 1962).

[17] J. N. Goldberg, Phys. Rev. **89**, 263 (1953).

[18] J. N. Goldberg, Phys. Rev. **111**, 315 (1958).

[19] P. G. Bergmann, Phys. Rev. **112**, 287 (1958).

[20] J. N. Goldberg, in *Les Théories Relativiste de la Gravitation, Royaumont 21–27 Juin, 1959* (Centre National de la Recherche Scientifique, Paris, 1962).

tational field. In the following sections this information will be used to construct asymptotically invariant surface integrals which may be related to the multipole structure of an isolated distribution of matter.

Before discussing the gravitational field, the electromagnetic field will be considered. This case is simpler and permits an explicit analysis of the field equations and the corresponding surface integrals. It will appear that those properties of the electromagnetic field which permit construction of asymptotically invariant surface integrals are analogous to the properties of the Riemann tensor which were used by Newman and Penrose as suitable conditions on outgoing gravitational radiation fields.[10] Accordingly, in Sec. 3, the corresponding integrals are constructed from the Riemann tensor. Finally, in Sec. 4, it will be shown that if $U_m{}^{ns}$ and $t_m{}^n$ have certain algebraic properties, then reasonable conditions exist for w^m such that the surface integrals (1.6) are satisfied asymptotically. For the Bondi metric[11] the superpotential and pseudotensor have the necessary algebraic properties.

Integrals of the form (1.5) are usually identified with the generators of the invariant transformations of the theory.[19,21] In general relativity these are the coordinate transformations. The integrals to be constructed here, however, are embedded in null hypersurfaces and, in general, they are not constants of the motion. As a result, their role as generators of canonical transformations is not clear. Therefore, in this paper only the construction of the integrals is undertaken, as their relation to the transformations requires further study.[22]

2. ELECTROMAGNETISM

Maxwell's equations for the electromagnetic field may be written in Minkowski space as

$$F^{ab}{}_{,b}+4\pi j^a\,,\quad F^{a*b}{}_{,b}=0\rightleftarrows F^{[ab,c]}=0\,,\qquad (2.1)$$
$$F^{a*b}=\tfrac{1}{2}\epsilon^{abcd}F_{cd}\,,$$

where j^a is the charge-current density, and ϵ^{abcd} is the totally skew tensor with $\epsilon_{1234}=-\epsilon^{1234}=1$. The gauge identity, $F^{ab}{}_{,ba}\equiv 0$, leads to the conservation of charge $j^a{}_{,a}=0$. If the field equations are multiplied by an arbitrary scalar function w, they may be rewritten as

$$(wF^{(-)ab})_{,b}-w_{,b}F^{(-)ab}=4\pi w j^a\,,$$
$$F^{(-)ab}\overset{\text{def}}{=}F^{ab}+iF^{a*b}.$$

Integration over a closed two-dimensional surface wholly enclosing the charge-current distribution yields surface integrals corresponding to (1.4):

$$\frac{d}{dx^0}\oint wF^{(-)0\alpha}n_\alpha dS=-\oint w_{,n}F^{(-)\alpha n}n_\alpha dS.\qquad (2.2)$$

[21] P. G. Bergmann and R. Schiller, Phys. Rev. **89**, 4 (1953).
[22] A. Komar, Phys. Rev. **127**, 955 (1962).

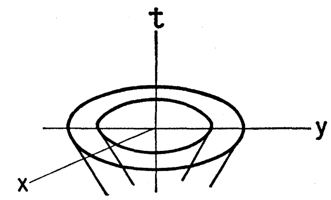

FIG. 1. The projection into the x-y plane of two concentric spheres, showing that they lie in different null cones.

The quantity

$$Q'[w]=(1/4\pi)\oint wF^{(-)0\alpha}n_\alpha dS$$

is a functional depending on the arbitrary function w. The question is whether w can be so chosen that Q' measures intrinsic properties of the charge distribution. Clearly, $Q'[1]$ is just the total charge. In order to describe other properties of the charge distribution and not of the electromagnetic field, one requires that Q' be independent of the particular closed surface; hence,

$$(wF^{(-)0\alpha})_{,\alpha}=0.\qquad (2.3)$$

From Eqs. (2.2) and (2.3) it is clear that the surfaces of integration being considered all lie in the space-like surface $x^0=\text{const}$. This problem has already been examined both in electromagnetic theory and in linearized gravitational theory.[16] One finds that a modification of Q' can be constructed which describes the essentially static part of the electric multipole moments. That only such information should be available from one hypersurface $x^0=\text{const}$ is reasonable. Two different concentric spherical surfaces embedded in the same hypersurface of constant x^0 lie in different null cones (Fig. 1). Therefore, if integrals taken over these surfaces are to describe the same physical property, all influence of the radiation field must be removed and only that part of the field which is determined on the hypersurface itself can be used to construct such properties.

To obtain information about the radiation field, hence, about the dynamical part of the multipole structure, it is necessary to consider surfaces of integration which lie in the same null cone. Then one can ask for properties which are unchanged as the surface of integration is slid along the null cone or distorted while it still lies in the same null cone (Fig. 2). However, in this case one must be able to distinguish the radiation field from the near field. Therefore, one can expect to obtain significant results only from the far, or asymptotic, field.

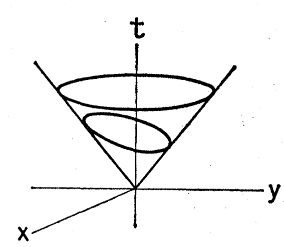

FIG. 2. The projection of different closed two-dimensional surfaces which lie in the same null cone.

The retarded electromagnetic field from a localized charge distribution has the asymptotic form[23,24]

$$F^{(-)ab}=\frac{{}_0N^{(-)ab}}{r}+\frac{{}_0\mathrm{III}^{(-)ab}}{r^2}+O(r^{-3}), \qquad (2.4)$$

where ${}_0N^{(-)ab}$ and ${}_0\mathrm{III}^{(-)ab}$ are skew tensors independent of r and with the important property that there exists a null vector k^a such that

$${}_0N^{(-)ab}k_b=0, \quad {}_0\mathrm{III}^{(-)ab}k_b=\alpha k^a, \quad k^ak_a=0. \qquad (2.5)$$

The subscript "0" to the left means that the quantities are constant along the null rays with tangent vecor k^a. This null vector is the propagation vector for the electromagnetic field in the wave zone. Furthermore, it is hypersurface orthogonal and, therefore, defines a family of null surfaces $u=$ const, $k_a=u_{,a}$. As suitable surfaces of integration in Eq. (2.2), closed two-dimensional surfaces, wholly contained in these null hypersurfaces, will be considered. Therefore, Eq. (2.2) may be rewritten in a covariant manner:

$$\frac{d}{du}\oint wF^{(-)ab}k_{[a}n_{b]}dS=-\oint w_{,b}F^{(-)ab}n_a dS. \qquad (2.2')$$

The condition that the asymptotic integral

$$Q[w]=\lim_{r\to\infty}\left(\frac{1}{4\pi}\right)\oint wF^{(-)ab}k_{[a}n_{b]}dS \qquad (2.6)$$

shall be independent of the two-surface embedded in $u=$ const becomes, in place of (2.3),

$$(wF^{(-)ab})_{,b}k_a=w_{,b}F^{(-)ab}k_a=O(r^{-4}). \qquad (2.7)$$

The current-free field equations have been used. From Eqs. (2.4) and (2.5), this condition becomes

$$-\left(\frac{\alpha}{r^2}\right)w_{,b}k^b+w_{,b}O(r^{-3})=O(r^{-4}).$$

The requirements on w implied by the above relation are clearly

$$w_{,a}=O(r^{-1}), \qquad (2.8a)$$

$$w_{,a}k^a=O(r^{-2}). \qquad (2.8b)$$

These requirements may be satisfied by choosing w to be a function only of suitably defined angular coordinates on the null surfaces.

In order to identify the quantities $Q[w]$, an explicit representation of the electromagnetic field is needed. This calculation is carried out in Appendix 1; the notation used in the following discussion is taken from there.

The surface of integration is defined by $u=$ const and $r=$ const. Therefore, the normals to the surface are k_a and $n_a=r_{,a}=-l_a$. From Eq. (A8) in the Appendix one finds

$$F^{(-)ab}k_an_b=\frac{1}{r^2}\Bigg[q+\sum_{N=0}^{\infty}(N+2)\frac{d^{N+1}}{du^{N+1}}Q^{ab:(r_N)}v_al_b(l_{r_N})$$
$$-i\sum_{N=0}^{\infty}(N+2)\frac{d^{N+1}}{du^{N+1}}Q^{a^*b:(r_N)}v_al_b(l_{r_N})\Bigg]+O(r^{-3}).$$

The collective indices are defined in Eq. (A4). Substituting the above expression into Eq. (2.6) one has for $w=1$,

$$Q[1]=q.$$

Define

$$w\overset{\text{def}}{=}w_N=\omega^{(s_N)}(l_{s_N}), \qquad (2.9)$$

where $\omega^{(s_N)}$ is a set of constants with the properties

$$\omega^{(s_{N-1})t}v_t=\omega^{(s_{N-2})t\cdot}{}_t=0$$

and

$$\omega^{s_1\cdots s_N}=\omega^{(s_1\cdots s_N)}.$$

The round brackets imply complete symmetrization in the indices enclosed. Thus, one has

$$Q[w_N]=(-1)^N\frac{(N+1)!}{\mathfrak{N}(N)}v_a\frac{d^N}{du^N}$$
$$\times[Q^{ar_N:(r_{N-1})}-iQ^{a^*r_N:(r_{N-1})}]\omega_{(r_N)}.$$

Thus, for appropriately chosen weighting functions, as defined by Eq. (2.9), $Q[w]$ measures certain time derivatives of the electric and magnetic multiple moments. From Eq. (2.2′) one sees that the rate of change of $Q[w]$ is determined by the null part of the asymptotic field ${}_0N^{(-)ab}$. This result is interesting because it is the null part which contributes to the energy-momentum tensor in the wave zone and, thus, determines the energy radiated by the system. In the absence of electromagnetic radiation, all the quantities $Q[w]$ are constants of the motion. However, in all cases $Q[1]$ is a constant of the motion, as follows from (2.2′). This, of course, is merely a restatement of the law of conservation of charge.

3. THE RIEMANN TENSOR

In a remarkable paper,[5] Sachs has carried out a penetrating analysis of the propagation of the Riemann tensor along geodesic rays. According to Sachs, *a vacuum metric, $R_{ab}=0$, has geodesic rays if there exists a vector field k'^a which is tangent to a congruence of null geodesics and satisfies the algebraic condition*[6,7]

$$k'_{[a}R_{b]ij[c}k'_{d]}k'^ik'^j=0. \qquad (3.1)$$

When geodesic rays exist, Sachs shows that (except in certain degenerate cases which are not of interest here)

[23] R. K. Sachs, in *Recent Developments in General Relativity* (Państwowe Wydawnictwo Naukowe, PWN—Polish Scientific Publishers, Warsaw, and Pergamon Press, Inc., London, 1962).
[24] J. N. Goldberg and R. P. Kerr (unpublished).

the Riemann tensor has the following expansion:

$$R^{abcd} = \frac{{}_0N^{abcd}}{r} + \frac{{}_0\text{III}^{abcd}}{r^2} + \frac{{}_0\text{II}^{abcd}}{r^3} + \frac{{}_0\text{I}^{abcd}}{r^4} + \cdots, \quad (3.2)$$

where r is an affine parameter along the null rays. Each of the numerators possesses all of the symmetries of a vacuum Riemann tensor. Therefore, each belongs to a particular Petrov class (with respect to the *same* null vector k'^a) which is indicated by ${}_0N$, ${}_0\text{III}$, ${}_0\text{II}$, and ${}_0\text{I}$; the subscripts "0" indicate that the numerators are constant along the ray direction, as before.

In general, a metric cannot be expected to have geodesic rays. A somewhat weaker requirement is that one of the null directions satisfying Eq. (3.1) be asymptotically geodesic; that is,

$$k'_{a;b}k'^b = O(r^{-n}), \quad n \geq 2.$$

Sachs conjectured that under these conditions the expansion (3.2) still holds, but k'^a is no longer a ray vector for the numerators. [A ray vector is a null vector satisfying Eq. (3.1)]. However, in keeping with the notion of a space with asymptotically geodesic rays there must exist a congruence of null geodesics with tangent vector k^a which represent the asymptotes for the rays k'^a. Furthermore, if the behavior of a space-time with geodesic rays is to be realized asymptotically, one may expect that *k^a is the ray vector for the first four numerators, but not for the fifth*; that is,

$$k_{[a}R_{a]ij[c}k_{d]}k^ik^j = O(r^{-5}). \quad (3.3)$$

Newman and Penrose[10] have shown that if there exists a hypersurface-orthogonal ray congruence satisfying Eq. (3.3), then the expansion (3.2) follows. Only such fields will be considered here. Therefore, in the following, the null vector satisfying (3.3) is a gradient

$$k_a = u_{,a} \quad (3.4)$$

and $u=$ const defines a family of null hypersurfaces.

From the properties[25]

$${}_0N^{abcd}k_d = 0, \quad {}_0\text{III}^{abcd}k_d = k^cL^{ab}, \quad (3.5)$$

one sees that the Riemann tensor has similar asymptotic properties to those required for the definition of asymptotic invariants for the electromagnetic field. How about the field equations? From the Bianchi identities one has

$$R^{ab[cd;e]} \equiv 0 \rightleftarrows R^{abc*d}{}_{;d} \equiv 0,$$

$$R^{abc*d} = \tfrac{1}{2}\epsilon^{cdij}R^{ab}{}_{ij}, \quad \epsilon_{1234} = \sqrt{-g}.$$

When $R_{ab}=0$, the first of the above equations reduces to

$$R^{abcd}{}_{;d} = 0.$$

These relations may be written succinctly as[26]

$$R^{(-)abcd}{}_{;d} = 0,$$
$$R^{(-)abcd} = R^{abcd} + iR^{abc*d}. \quad (3.6)$$

Thus, $w_{ab}R^{(-)abcd}$, w_{ab}, an arbitrary bivector, plays the same role as $wF^{(-)ab}$ in electromagnetism. From the field equations (3.6) one can construct the integral relations corresponding to Eq. (2.2′),

$$\frac{d}{du}\oint w_{ab}(-g)^{1/2}R^{(-)abcd}u_{,c}n_d dS = \oint w_{ab;c}(-g)^{1/2}R^{(-)abcd}n_d dS. \quad (3.7)$$

The requirement that

$$M[w_{ab}] \overset{\text{def}}{=} \lim_{r\to\infty} \oint w_{ab}(-g)^{1/2}R^{(-)abcd}u_{,c}n_d dS \quad (3.8)$$

be independent of the surface of integration, as long as it is a space-like section of $u=$ const, leads to the condition

$$w_{ab;d}R^{(-)abcd}k_c = O(r^{-4}),$$

where Eqs. (3.4) and (3.6) have been used. From the asymptotic behavior given in Eq. (3.2) and the properties (3.5), this condition becomes

$$w_{ab;d}\left[k^d\frac{L^{ab}}{r^2} + O(r^{-3})\right] = O(r^{-4}).$$

Thus, the requirements imposed on w_{ab}, corresponding to those of Eq. (2.8), are

$$w_{ab;d} = O(r^{-1}), \quad (3.9a)$$

$$w_{ab;d}k^d = O(r^{-2}). \quad (3.9b)$$

These conditions may easily be satisfied by choosing w_{ab} to be a function of suitably defined angular coordinates, as was done in the previous section for the weight function, w.

In Appendix 2 the linear weak field approximation to the Einstein field equations is considered. From Eq. (A16) one sees that all the asymptotic information may be obtained with the weighting tensor w_{ab} chosen to be

$$w_{ab} \overset{\text{def}}{=} w_{Mab} = l_{[a}\omega_{b]:}{}^{(s_M)}(l_{s_M}), \quad (3.10)$$

with $\omega_{b:}{}^{(s_M)}$ a set of constants with the same properties on the indices s_i as in Eq. (2.9) for $\omega^{(s_M)}$ (the colon is used to emphasize this relationship).

[25] R. K. Sachs, Z. Physik **157**, 1462 (1960).

[26] J. Geheniau and R. Debever, Bull. Classe Sci., Acad. Roy. Belg. **42**, 252 (1956).

With this choice one obtains from Eq. (3.8)

$$M[W_{Mab}]=-2\omega_{b:}{}^{(sM)}\oint\sum_{N=0}^{\infty}(N+4)I^b{}_a\frac{d^{N+3}}{du^{N+3}}[M^{aijn:(r_N)}-iM^{aij*n:(r_N)}]\times v_nk_ik_j(l_{r_N})(l_{sM})\sin\theta d\theta d\varphi. \quad (3.11)$$

A study of the integrand of the above equation, together with the integral relations given in Appendix 1, shows that for a given M, one gets contributions from $M^{aibj:(r_N)}$ with $N=M-1$, $M-2$, and $M-3$. Therefore, these integrals do not produce a clear separation of the multipole moments as was true in the electromagnetic case. However, the third time derivatives of the quadrupole moment is singled out by $M=1$. To show the complexity of these integrals, Eq. (3.11) will be evaluated for the case where only a quadrupole moment exists; that is, the sum under the integral sign contains only the term $N=0$.

$$M[w_{1ab}]=\frac{8}{5}\omega_{b:s}\frac{d^3}{du^3}[M^{bijm}-iM^{bij*m}]v_iv_j, \quad (3.12a)$$

$$M[w_{2ab}]=-\frac{16}{15}\omega_{b:rs}\frac{d^3}{du^3}[M^{i(rs)j}-iM^{i(rs)*j}]v_j\epsilon_i{}^b, \quad (3.12b)$$

$$M[w_{3ab}]=\frac{16}{35}\omega_{b:}{}^b{}_{.rs}\frac{d^3}{du^3}[M^{risj}-iM^{ris*j}]v_iv_j. \quad (3.12c)$$

Clearly, (3.12c) can always be made to vanish by choosing $\omega_{b:}{}^b{}_{.rs}=0$. Since the contraction of any pair of indices on the multipole moments vanishes, without loss of information, one can impose the additional condition

$$w_{b:}{}^{b(sM-1)}=0, \quad (3.13)$$

Eq. (3.12b), on the other hand, is related to the dual of (3.12a). Therefore, by algebraic means one can eliminate this contribution. It is easy to see that all integrals involving higher multipoles will have a similar structure.

Therefore, in those space-times in which the Riemann tensor has the expansion (3.2) and the metric tensor approaches the Minkowski metric as r^{-1}, a physical interpretation for the integrals (3.8) exists. Certainly, the existence of the expansion (3.2) for the Riemann tensor does not guarantee that the space-time admits a metric which is asymptotically Minkowskian. For example, the Robinson-Trautman metrics do not have this asymptotic behavior except for the type D metrics.[27] It is not yet clear, however, whether an interpretation can be given for these cases as well.

It is interesting to note that the time derivatives which appear above are of the *same* order as those which contribute to the energy transport as calculated by the Einstein pseudotensor,[28] whereas the time derivatives which appeared in the electromagnetic example are of one order lower than those which appear in the energy transport as calculated by the Maxwell stress-energy tensor. This results, of course, from the fact that the analogy between the Riemann tensor and the electromagnetic field is not complete. R^{abcd} involves second derivatives of the gravitational potentials (the metric tensor) whereas F^{ab} involves only first derivatives of the vector potential. In both cases, however, energy is calculated by an expression which is quadratic in first derivatives of the corresponding potentials. The Maxwell stress-energy tensor is homogeneous-quadratic in the F^{ab}; thus, its coefficient of r^{-2} arises from the *asymptotic null field*. The Einstein pseudotensor, on the other hand, is an expression which is *linear* in the Riemann tensor and from which the second derivatives have been removed by means of the superpotential.[18,20] *Thus, the coefficient of r^{-2} in the pseudotensor, which describes the radiation of energy, necessarily depends on the asymptotic type* III *field and not the null field.*

However, the asymptotic null field is important for if it vanishes the quantities $M[w_{ab}]$ will be constants of the motion. This result follows easily from Eqs. (3.8a) and the conditions in (3.10). A vanishing null field, therefore, implies that energy is being radiated at a constant rate. If the constant rate is zero, that is, no energy is radiated, the quantities $M[w_{ab}]=0$.

The null field is also important for the existence of the superenergy tensor defined by Bel and Robinson.[6,29,30] This tensor is quadratic in the Riemann tensor and, therefore, is closer to the electromagnetic analogy than is the pseudotensor. The structure of the Bel-Robinson tensor has been discussed in some detail by Trautman[31] who concludes that it is not suitable as an energy tensor. However, from the discussion in the previous paragraph, it is clear that *the Bel-Robinson tensor is related to the rate of change of the radiation field, though not to the radiation field itself.*

4. THE SUPERPOTENTIAL

The success in the construction of asymptotically invariant integrals with the Riemann tensor prompts one to inquire about the conditions under which the integrals of Eq. (1.5) may also be asymptotically invariant. Therefore, define

$$U[w^m]\overset{\text{def}}{=}\lim_{r\to\infty}\frac{1}{16\pi}\oint w^mU_m{}^{[ns]}k_{[n}n_{s]}dS, \quad (4.1)$$

where r is a suitable parameter defined along a family of

[27] I. Robinson and A. Trautman, Proc. Roy. Soc. (London) **A265**, 463 (1962).

[28] J. Boardman and P. G. Bergmann, Phys. Rev. **115**, 1318 (1959).

[29] L. Bel, Compt. Rend. **248**, 1297 (1959).

[30] I. Robinson, in *Les Théories Relativiste de la Gravitation, Royaumont 21–27 Juin, 1959* (Centre National de la Recherche Scientifique, Paris, 1962).

[31] A. Trautman, in *Gravitation,* edited by L. Witten (John Wiley & Sons, Inc., New York, 1962).

null geodesics which generate a null surface asymptotically, in the sense of Eq. (3.5). Asymptotic invariance then requires that

$$(w^m U_m{}^{[ns]})_{,s} k_n = O(r^{-4})\,.$$

Using the field equations $G_a{}^b=0$ and Eq. (1.2), the above condition becomes

$$w^m{}_{,s} U_m{}^{[ns]} k_n + w^m t_m{}^n k_n = O(r^{-4})\,. \tag{4.2}$$

From the previous integrals constructed, one sees that

$$w^m = O(r^0)\,, \tag{4.3a}$$

$$w^m{}_{,s} = O(r^{-1})\,, \tag{4.3b}$$

$$w^m{}_{,s} k^s = O(r^{-2}) \tag{4.3c}$$

are reasonable conditions to impose on the weighting vector w^m. Assuming (4.3), one finds that (4.2) requires

$$U_m{}^{[ns]} k_n = r^{-2}{}_0A_m k^s + O(r^{-3})\,, \tag{4.4a}$$

$$t_m{}^n k_n = O(r^{-4})\,. \tag{4.4b}$$

From Eq. (A18) in the Appendix one sees that condition (4.4a) is satisfied in the linearized theory. Condition (4.4b) is automatically satisfied there since $t_m{}^n$ contains no linear terms.

Trautman[4,23] has shown that when w^m is constant the integral (4.1) *exists* with the assumption of certain outgoing boundary conditions; namely,

$$g_{ab} = \eta_{ab} + O(r^{-1})\,,$$

$$g_{ab,c} = i_{ab} k_c + O(r^{-2})\,, \quad i_{ab} = O(r^{-1})\,, \quad k^a k_a = 0\,,$$

$$(i_{ab} - \tfrac{1}{2}\eta_{ab} i) k^b = O(r^{-2})\,, \quad i = \eta^{ab} i_{ab}\,.$$

These conditions, however, are not strong enough to establish (4.4).

Bondi and his co-workers[11] have constructed an asymptotic metric with the following assumptions:

(1) There exists a family of null surfaces, $u=$const, which is generated by a congruence of expanding null geodesics.

(2) Each null ray lying in a given surface $u=$const is uniquely defined by spherical angular coordinates θ and ϕ.

(3) One chooses a coordinate r along the null geodesics such that the area element of the two-dimensional surfaces $u=$const, $r=$const is simply

$$dS = r^2 \sin\theta d\theta d\varphi\,.$$

(4) All relevant physical or geometrical quantities possess an expansion in r^{-1}; this statement contains essentially the restriction to outgoing radiation.[32]

With the further simplification of cylindrical symmetry, Bondi writes the metric as

$$dS^2 = \left(e^{2\beta}\frac{V}{r} - e^{2\gamma} r^2 U^2\right) du^2 + 2l^{2\beta} du dr + 2e^{2\gamma} r^2 U du d\theta$$
$$- r^2 (e^{2\gamma} d\theta^2 + e^{-2\gamma} \sin^2\theta d\varphi^2)\,, \tag{4.5}$$

where

$$\gamma = c(u,\theta)/r + \cdots\,,$$
$$U = r^{-2}(c_{,2} + 2c \cot\theta) + \cdots\,,$$
$$V = r - 2M(u,\theta)\,,$$
$$\beta = -c^2/4r^2 + \cdots\,.$$

Bondi refers to $c_{,0}(u,\theta)$ as the *news function.* It is completely arbitrary except for its behavior in the neighborhood of $\theta=0, \pi$:

$$c(u,\theta)\,|_{\theta=0,\pi} = 0(\sin^2\theta)\,.$$

The function $M(u,\theta)$ is closely related to the mass; indeed, for the Schwarzschild metric, M is the mass.

There is one difficulty with the metric in the form (4.5). The pseudotensor $t_m{}^n$ and, hence, the superpotential, has been shown to be meaningful only when the coordinates are asymptotically rectangular.[33] However, one can avoid transforming the metric by the following ruse. From Eq. (1.3a) it is clear that $U_m{}^{ns}$ is homogeneous linear in the first derivatives of the metric. These first derivatives may be expressed in terms of the Christoffel symbols as[34]

$$g_{ab,c} = [ca,b] + [cb,a] = g_{bi}\begin{Bmatrix} i \\ ac \end{Bmatrix} + g_{ai}\begin{Bmatrix} i \\ bc \end{Bmatrix}.$$

A brief calculation then gives

$$U_m{}^{[ns]} = (-g)^{1/2} g_{mk}\left[\begin{Bmatrix} n \\ tr \end{Bmatrix} g^{stkr} + \begin{Bmatrix} s \\ tr \end{Bmatrix} g^{tnkr} + \begin{Bmatrix} t \\ tr \end{Bmatrix} g^{nskr}\right], \tag{4.6}$$

$$g^{stkr} = g^{sk} g^{tr} - g^{sr} g^{tk}\,.$$

The pseudotensor is already expressed in terms of the Christoffel symbols in Eq. (1.3b).

The transformation properties of $U_m{}^{[ns]}$ and $t_m{}^n$ are determined by those of the Christoffel symbols, which are not tensorial. However, the difference between two different sets of Christoffel symbols will transform as a tensor.[34] Assume that x^m are asymptotically rectangular coordinates; that is, the metric tensor is asymptotically Minkowskian in a Cartesian frame. Consider the transformation to any other system of coordinate $x^{m'}$

$$x^{m'} = x^{m'}(x^m)\,, \tag{4.7}$$

which need no longer be asymptotically rectangular.

[32] R. K. Sachs, Proc. Roy. Soc. (London) (to be published).

[33] A. Einstein, Berlin Ber. 448 (1918).

[34] P. G. Bergmann, *Introduction to the Theory of Relativity* (Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1947).

The Christoffel symbols transform as follows:

$$\left\{\begin{matrix} m' \\ n'j' \end{matrix}\right\} = A^{n}{}_{n'}A^{j}{}_{j'}A^{m'}{}_{m}\left\{\begin{matrix} m \\ nj \end{matrix}\right\} + A^{m'}{}_{m}A^{m}{}_{j',n'},$$

$$A^{m'}{}_{m}=\frac{\partial x^{m'}}{\partial x^{m}}, \quad A^{m}{}_{m'}=\frac{\partial x^{m}}{\partial x^{m'}}. \tag{4.8}$$

Apply this transformation to Minkowski space. In the Cartesian frame,

$$\left\{\begin{matrix} m \\ nj \end{matrix}\right\}_0 = 0.$$

However, in the transformed frame (4.7),

$$\left\{\begin{matrix} m' \\ n'j' \end{matrix}\right\}_0 = A^{m'}{}_{m}A^{m}{}_{j',n'}. \tag{4.9}$$

Taking the difference between (4.8) and (4.9) one has that the difference

$$\left\{\begin{matrix} m \\ nj \end{matrix}\right\}^{*} \overset{\text{def}}{=} \left\{\begin{matrix} m \\ nj \end{matrix}\right\} - \left\{\begin{matrix} m \\ nj \end{matrix}\right\}_0 \tag{4.10}$$

transforms as a tensor and in the asymptotically rectangular coordinate frame x^m, one has simply

$$\left\{\begin{matrix} m \\ nj \end{matrix}\right\}^{*} = \left\{\begin{matrix} m \\ nj \end{matrix}\right\}.$$

Therefore, maintaining the distinction between primed and unprimed coordinates introduced above Eq. (4.7), the superpotential and pseudotensor in the asymptotically rectangular coordinate system may be written

$$U_m{}^{[ns]} = A^{m'}{}_{m}A^{n}{}_{n'}A^{s}{}_{s'}U_{m'}{}^{[n's']*}, \tag{4.11a}$$

$$t_m{}^{n} = A^{m'}{}_{m}A^{n}{}_{n'}t_{m'}{}^{n'*}, \tag{4.11b}$$

where the superscript asterisk means that the substitution defined by Eq. (4.10) is made in (1.3b) and (4.6).

The transformation involved for the Bondi coordinates requires removing the null coordinate $u = x^0 - r$ and transforming from polar coordinates to rectangular coordinates. If $x^{m'} = (u,r,\theta,\phi)$, then

$$\begin{aligned} x^0 &= u + r, \\ x^1 &= r\sin\theta\cos\varphi, \\ x^2 &= r\sin\theta\sin\varphi, \\ x^3 &= r\cos\theta. \end{aligned} \tag{4.12}$$

Thus,

$$A^{m'}{}_{m} = \begin{bmatrix} 1 & -\sin\theta\cos\varphi & -\sin\theta\sin\varphi & -\cos\theta \\ 0 & \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ 0 & \cos\theta\cos\varphi/r & \cos\theta\sin\varphi/r & -\sin\theta/r \\ 0 & -\sin\varphi/r\sin\theta & \cos\varphi/r\sin\theta & 0 \end{bmatrix}, \tag{4.13a}$$

$$A^{m}{}_{m'} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ 0 & \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\cos\theta\cos\varphi \\ 0 & \cos\theta & -r\sin\theta & 0 \end{bmatrix}. \tag{4.13b}$$

With the help of these relations it is an easy, though somewhat laborious, matter to prove that (i) the Bondi metric satisfies the Trautman boundary conditions[35] and (ii) the requirements listed in Eq. (4.4) are satisfied. Therefore, with the weighting vector limited by(4.3), the integrals defined by (4.1) are asymptotically invariant.

Although one knows from Trautman's work[31] that when w^m is constant the integrals (4.1) define the total energy and momentum of the system, it is instructive to examine the linear approximation to see how the multipoles come out. Using Eq. (A18) from the Appendix, one has that

$$U[w_M{}^m] = -4\omega^{t:(s_M)}\oint(l_{s_M})\Big[mv_t + \sum_{N=0}^{\infty}\frac{d^{N+2}}{du^{N+2}} \times M_t{}^{inj:(r_N)}k_iv_nl_j(l_{r_N})\Big]\sin\theta d\theta d\varphi, \tag{4.14}$$

where

$$w_M{}^m = \omega^{m:(s_M)}(l_{s_M});$$

the constants $\omega^{m:(s_M)}$ have all the symmetries in the indices (s_M) as listed in Eq. (2.9). Carrying out the integration explicitly with the help of (A4), one finds

$$U[w_M{}^a] = \frac{(-1)^{M+1}M!}{\mathfrak{N}(M)}\omega_{a:bc(s_{M-2})} \times\Big[(M+1)\frac{d^M}{du^M}M^{abnc:(s_{M-2})}v_n + (M+2)\frac{d^{M+1}}{du^{M+1}}M^{aijb:c(s_{M-2})}v_iv_j\Big]. \tag{4.15}$$

$$U[w_0{}^a] = -\omega_a mv^a,$$

$$U[w_1{}^a] = \tfrac{1}{3}\omega_{a:b}\frac{d^2}{du^2}M^{aijb}v_iv_j.$$

In particular, one has $U[w_0{}^a]$, as expected, contains the total energy and momentum of the system; $U[w_1{}^a]$ contains only the vibrational quadrupole moment; but in general, $U[w_M{}^a]$ contains both the vibrational and rotational multipole moments of order 2^M and the vibrational moment of order 2^{M+1}: These vibrational moments may be separated out by considering $U[w_{M+1}{}^a]$ with $\omega_{a:(s_M+1)} = v_a\omega_{(s_M+1)}$. It is fortunate that this separation can be carried out at all, for certainly the significance of the dual of the superpotential is unclear, even though it does satisfy a conservation law.

[35] F. A. E. Pirani (private communication).

For the Bondi metric one finds that

$$
\begin{aligned}
U_m{}^{[ns]}k_n r_{,s}
&= -r^{-2}A^{m'}{}_m\{\delta^0{}_{m'}[-4M+(1/\sin\theta)\\
&\qquad \times(c_{,2}\sin\theta+2c\cos\theta)_{,2}]\\
&\qquad -\delta^2{}_{m'}r(c_{,2}+2c\cot\theta)\}+O(r^{-3}).
\end{aligned}
\tag{4.16}
$$

Therefore, for the invariant integrals (4.1), one has

$$
U[w_0{}^a]=\frac{1}{4\pi}\oint[\omega^0M-\omega^3M\cos\theta]\sin\theta d\theta d\varphi\,,
$$

$$
\begin{aligned}
U[w_1{}^a]=\frac{1}{8\pi}\oint\{&\omega^0{}_{:3}2M\cos\theta-\omega^1{}_{:1}(2M-c)\sin^2\theta\cos^2\varphi\\
&-\omega^2{}_{:2}(2M-c)\sin^2\theta\sin^2\varphi\\
&-\omega^3{}_{:3}(2M\cos^2\theta-c\sin^2\theta)\}\sin\theta d\theta d\varphi.
\end{aligned}
$$

The higher multipole moments are obtained by weighting (4.16) with the appropriate spherical harmonics. Actually, it is not the multipole moments themselves, but various time derivatives of the moments, which are defined by these integrals. As in the case of the electromagnetic field, these time derivatives are each one lower than those which determine the energy transport by means of the pseudotensor. The additional time derivative results from the time dependence of M and c.The rate of change of the quantities defined by U is given by the right-hand side of Eq. (1.4) in the limit of $r\rightarrow\infty$. In particular, when $w^m=\omega^m=$ const, one has the energy-momentum transport given by

$$
\begin{aligned}
&\frac{1}{4\pi}\frac{d}{du}\oint[\omega^0M-\omega^3M\cos\theta]\sin\theta d\theta d\varphi\\
&\qquad=-\frac{1}{4\pi}\oint\left(\frac{dc}{du}\right)^2 k_a\omega^a\sin\theta d\theta d\varphi.
\end{aligned}
\tag{4.17}
$$

It is clear that the quantity which is identified with the total energy, the coefficient of ω^0, is necessarily a *decreasing function of time* as long as the news function $c_{,0}(u,\theta)$ does not vanish. Since c is independent of ϕ, the momentum components can change only along the axis of symmetry. However, it is clear that there need not be a change in the total momentum; for example, $c=f(u)\sin^2\theta$. One can show, furthermore, that all of the quantities defined by Eq. (4.1) are constants of the motion unless the news function does not vanish.

The most important of the above conclusions that concerning the energy, was already obtained by Bondi. He used the Bianchi identities to arrive at the "supplementary condition"

$$
2M_{,0}=-2(c_{,0})^2+(1/\sin\theta)[c_{,2}\sin\theta+2c\cos\theta]_{,2}.\tag{4.18}
$$

By multiplying this equation by $\sin\theta$ and integrating, he was able to conclude that if $c_{,0}\neq0$ during a finite interval, the mass, or energy of the system is less at the end of the interval. Clearly, this result is the same as that given in Eq. (4.17) for $\omega^a=\delta_0{}^a$ except that the interpretation here does not depend on the possible existence of initial and final rest states.

One cannot make such a general statement about the momentum. The momentum may increase, decrease, or remain constant, depending on the angular dependence of $c_{,0}$—hence, depending on the angular dependence of the energy transport. However, if $c_{,0}=0$, the linear momentum is a constant of the motion. Clearly, in this case the momentum may be reduced to zero by a Lorentz transformation.

Similarly, the Nth time derivative of the 2^N-pole components may increase, decrease, or be constants of the motion. If $c_{,0}=0$, they are necessarily constants of the motion. However, the time derivatives of lower order need not vanish. Indeed, these are just the *nonradiative* motions described by Bondi.[11] Unfortunately, this analysis sheds no light on their origin. However, it is clear from the results of Sec. 2 on electromagnetism that similar nonradiative motions occur there. In the electromagnetic case, moreover, one would not identify such nonradiative motions with a system of charges which interacts only through its own field. Bondi does suggest an interpretation for gravitational interactions as it is consistent with Infeld's conclusions about gravitational radiation.[36] The results of this paper do not rule out such a possibility.

A little thought shows that Eq. (4.1) considered for all possible vector functions w^a is merely an integral statement of (4.18). This integral form has the advantage of permitting a physical interpretation, as given above. Furthermore, the physical picture implied by the asymptotic integrals allows one to conceive of measurements made at large distances which give information about the source distribution. Only a sensitive meter for the gravitational field is lacking.

5. DISCUSSION

Thus, asymptotic integral invariants may be constructed either from the Riemann tensor or from the superpotential. In both cases physically important quantities result. They are different, yet closely related. From the Riemann tensor one is not led to the total energy or mass of the material system generating the gravitational field. One is led, however, to time derivatives of the multipole moments of the matter distribution. Specifically, one gets the $(N+1)$th time derivative of the 2^N pole. By examining the energy transport with Einstein pseudotensor, one sees that these time derivatives are precisely those which determine the energy flow. The result is curious, for these time derivatives do not come from the *far wave zone* of the Riemann tensor, that is, from the null field which falls off as r^{-1}, but from the *near wave zone*, the asymptotically type III part of

[36] L. Infeld and J. Plebanski, *Motion and Relativity* (Państwowe Wydawnictwo Naukowe, PWN—Polish Scientific Publishers, Warsaw and Pergamon Press, Inc., London, 1960).

the Riemann tensor, which falls off as r^{-2}. As was discussed in Sec. 3, the linear relationship between the Riemann tensor and the pseudotensor leads to this result. An examination of the Riemann tensor for Bondi's metric shows that one can construct a gravitational field in which the Riemann tensor falls off as r^{-2} and not as r^{-1}, that is, the null part of the field vanishes, yet there is an energy loss; in fact the rate at which energy is lost will be constant, in agreement with the previous discussion. However, such a solution seems to be ruled out because the *asymptotically flat* boundary condition is inconsistent with a constant rate of radiation.[37–39]

However, it is the superpotential which is closer to the physical ideas concerning energy and energy transport.[31] This quantity expresses the *strong conservation laws* which are a restatement of the Bianchi identities, from which Bondi's supplementary conditions are derived. As a result, from these conditions one gets precisely the same information that is obtained asymptotically with the superpotential, *and only that information.*

Actually, here only one of Bondi's two supplementary conditions has been considered. The other discusses a quantity which is identified with the dipole moment. This quantity is not examined by the integrals constructed here for it appears in the r^{-3} part of the Riemann tensor and the superpotential. Thus, one might expect it to appear in the study of angular momentum. A number of possible candidates for an angular momentum complex have been proposed.[18,40,41] These should be studied from the point of view presented in this paper. Undoubtedly, Bondi's second supplementary condition contains all this information; only the interpretation may be added.

The invariance of the asymptotic integrals remains to be discussed. Certainly they are invariant under all transformations which are asymptotically a homogeneous Lorentz transformation. Such transformations only vary the surface of integration in a given null surface and the integrals were constructed to be independent of such changes. However, they are clearly altered if the null surfaces are changed, for the integrals evaluate information which is transmitted along the null rays. Therefore, changing the null rays being examined changes the information being evaluated, and changes the surface integrals. Thus, the quantities defined are not invariant under the full Bondi-Metzner group.[11]

Finally, the possible relationship of these integrals to the generators of invariant transformations should be emphasized again. The fact that the superpotential and the pseudotensor generate canonical transformations has long been known.[7,21] Recently, Komar[22] has shown that integrals constructed by the Riemann tensor and a bivector as in Sec. 3 may generate infinitesimal coordinate transformations with the Jacobian equal to 1; that is, volume preserving transformations. As mentioned earlier, however, whether integrals formed on null hypersurfaces may be identified with generating functionals is still an open question.

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APPENDIX

1. The Electromagnetic Field

Assume that the charge distribution is confined within a sufficiently large world tube. Within the world tube take a straight world line $x^a = z^a(s)$, $ds^2 = \eta_{ab} dz^a dz^b$, where η_{ab} is the Minkowski metric $(1, -1, -1, -1)$. The four-velocity, or tangent vector to the world line, is $v^a = dz^a/ds = \dot{z}^a$.

Let $u(x)$ be the retarded solution for s of the equation

$$\eta_{ab}[x^a - z^a(s)][x^b - z^b(s)] = 0.$$

Clearly, $u(x) = \text{const}$ is a family of null surfaces. The following relations are easily established:

$$\dot{v}^a = 0, \quad k^a k_a = 0, \quad k^a v_a = -k^a l_a = 1,$$

with

$$\begin{aligned} k_a &= u_{,a} = \eta_{ab}[x^b - z^b(u)]/r, \\ r &= v_a[x^a - z^a(u)], \\ l_a &= k_a - v_a = -r_{,a}. \end{aligned} \tag{A1}$$

Furthermore, it is useful to introduce

$$\begin{aligned} k_{a,b} &= l_{a,b} = I_{ab}/r, \\ I_{ab} &= \eta_{ab} - v_a v_b + l_a l_b, \\ I_{ab,c} &= 2 l_{(a} I_{b)c}/r. \end{aligned} \tag{A2}$$

The quantities introduced are most easily identified in the rest frame $v^a = \delta_0{}^a$. One has then $u = x^0 - r$, and consequently

$$k_a = \delta_a{}^0 - \delta_a{}^s r_{,s}.$$

With the introduction of the usual spherical angles one has

$$l_a = -r_{,a} = (0, \sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta).$$

Defining the projection operator into the hypersurface

[37] A. Papapetrou, Ann. Physik **2**, 87 (1958).

[38] A. Peres and N. Rosen, in *Recent Developments in General Relativity* (Państwowe Wydawnictwo Naukowe, PWN—Polish Scientific Publishers, Warsaw and Pergamon Press, Inc., London, 1962).

[39] R. Arnowitt, S. Deser, and C. Misner, Phys. Rev. **121**, 1556 (1961).

[40] L. Landau and E. Lifshitz, *The Classical Theory of Fields* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1951).

[41] P. G. Bergmann and R. Thomson, Phys. Rev. **89**, 400 (1953).

$v_a dx^a=0$, $\epsilon_{ab}=\eta_{ab}-v_a v_b$, one can show that the following surface integral relations hold ($d\Omega=\sin\theta d\theta d\phi$):

$$\frac{1}{4\pi}\oint l_a l_b d\Omega=-(1/3)\epsilon_{ab}\,,$$

$$\frac{1}{4\pi}\oint l_a l_b l_c l_d d\Omega=\frac{1}{15}[\epsilon_{ab}\epsilon_{cd}+\epsilon_{ac}\epsilon_{bd}+\epsilon_{ad}\epsilon_{bc}]\,, \qquad \text{(A3a)}$$

$$\vdots$$

$$\frac{1}{4\pi}\oint l_{a_1}\cdots l_{a_{2N}}d\Omega=\frac{(-1)^N}{\mathfrak{N}(N)}\sum[\epsilon_{a_1a_2}\epsilon_{a_3a_4}\cdots\epsilon_{a_{2N-1}}\epsilon_{a_{2N}}].$$

The sum is taken over all symmetric combinations of the indices and

$$\mathfrak{N}(N)\overset{\text{def}}{=}(2N+1)(2N-1)\cdots 3\times 1\,.$$

Finally,

$$\frac{1}{4\pi}\oint l_{a_1}\cdots l_{a_{2N+1}}d\Omega=0\,. \qquad \text{(A3b)}$$

The following shorthand notation will be used hereafter:

$$(l_{r_N})=l_{r_1}l_{r_2}\cdots l_{r_N};$$
$$A^{(r_N)}=A^{r_1r_2\cdots r_N}. \qquad \text{(A4)}$$

Introducing the vector potential A^a such that

$$F^{ab}=A^{a,b}-A^{b,a}\,, \qquad \text{(A5)}$$

the field equations, (2.1) become

$$\Box A^a=4\pi j^a\,,\quad A^a{}_{,a}=0\,.$$

Take the retarded solution in the following form[42]:

$$A^a=\frac{qv^a}{r}+\sum_{N=0}^{\infty}\left\{\frac{Q^{as:(r_N)}(u)}{r}\right\}_{,s(r_N)}\,. \qquad \text{(A6)}$$

The coefficients $Q^{as:(r_N)}$ may be chosen to have the following properties:

$$Q^{as:(r_N)}=-Q^{sa:(r_N)}\,,$$
$$Q^{as:(r_N)}=Q^{as:(r_1r_2\cdots r_N)}\,,$$
$$Q^{as:(r_{N-1})t}v_t=0\,,$$
$$Q^{as:(r_{N-2})jk}\eta_{jk}=0\,,$$
$$Q^{as:(r_{N-1})t}\eta_{st}=0\,,$$
$$Q^{[as:t](r_{N-1})}=0\,. \qquad \text{(A7)}$$

One can show that

$$Q^{as:(r_N)}v_s \quad\text{and}\quad Q^{a^*s:(r_N)}v_s$$

represent the electric and magnetic multipoles, respectively, of order 2^{N+1}. The asterisk means the dual as defined in Eq. (2.1).

By means of a somewhat tedious calculation, one obtains the electromagnetic field from (A5):

$$F^{ab}=\frac{{}_0N^{ab}}{r}+\frac{{}_0\mathrm{III}^{ab}}{r^2}+O(r^{-3})\,, \qquad \text{(A8)}$$

with

$${}_0N^{ab}=2\sum_{N=0}^{\infty}k^{[b}\frac{d^{N+2}}{du^{N+2}}Q^{a]s:(r_N)}k_s(l_{r_N})\,,$$

$${}_0\mathrm{III}^{ab}=2qv^{[a}l^{b]}+2\sum_{N=0}^{\infty}(l_{r_{N-1}})\bigg[(N+1)\left(\frac{Nk_s}{2}+l_s\right)l_{r_N}k^{[b}$$
$$+k_s l_{r_N}m^{[b}+l_{r_N}I_s{}^{[b}+Nk_sI_{r_N}{}^{[b}\bigg]\frac{d^{N+1}}{du^{N+1}}Q^{a]s:(r_N)}\,.$$

Clearly

$${}_0N^{ab}k_b=0\,,$$

$${}_0\mathrm{III}^{ab}k_b$$
$$=-k^a\left[q+\sum_{N=0}^{\infty}(N+2)\frac{d^{N+1}}{du^{N+1}}Q^{bs:(r_N)}v_b l_s(l_{r_N})\right]. \qquad \text{(A9)}$$

Similar relations hold for the dual; one obtains

$${}_0N^{a^*b}k_b=0\,,$$

$${}_0\mathrm{III}^{a^*b}k_b=k^a\sum_{N=0}^{\infty}(N+2)\frac{d^{N+1}}{du^{N+1}}Q^{b^*s:(r_N)}v_b l_s(l_{r_N})\,. \qquad \text{(A10)}$$

In order to obtain (A8) easily, the following relation is used. Let $P^{M:(r_N)}(u)$ be a set of functions which is totally symmetric in the indices (r_N) and such that $P^{M:(r_{N-1})t}v_t=0$, then

$$(P^{M:(r_N)})_{,(r_N)}=\sum_{K=0}^{N}A_{NK}(l_{r_N})r^{-1-K}\frac{d^{N-K}}{du^{N-K}}P^{M:(r_N)}\,,$$
$$A_{NK}=(N+K)!/K!(N-K)!2^K. \qquad \text{(A11)}$$

2. Linear General Relativity

Introducing the deviation from flat space as

$$\sqrt{-g}g^{ab}=\eta^{ab}-\gamma^{ab}\,,$$

the linear terms in Einstein's field equations take the form

$$\eta^{rs}\gamma^{ab}{}_{,rs}=0\,,\quad \gamma^{ar}{}_{,r}=0 \qquad \text{(A12)}$$

outside the matter distribution. In the same manner as was done for the electromagnetic field, a solution for the γ^{ab} with outgoing waves only may be given in terms of a multipole expansion[43]

$$\gamma^{ab}=4\left\{\frac{mv^av^b}{r}+\left(\frac{S^{ab:r}}{r}\right)_{,r}+\sum_{N=0}^{\infty}\left[\frac{M^{aibj:(r_N)}}{r}\right]_{,ij(r_N)}\right\}\,, \qquad \text{(A13)}$$

[42] L. Lyuboshitz, and Ya. A. Smorodinskii, Zh. Eksperim. i Teor. Fiz. **42**, 846 (1962) [translation: Soviet Phys.—JETP **15**, 589 (1962)].

[43] R. K. Sachs and P. G. Bergmann, Phys. Rev. **112**, 674 (1958).

The quantities $M^{aibj:(r_N)}$ have all the properties of a vacuum Riemann tensor in the first four indices $(aibj)$; in addition, they have all the properties listed in Eq. (A7) for the electromagnetic multipole moments. The remaining quantities in Eq. (A13) have the following properties:

$$\dot{m}=0,$$
$$S^{mn:r}=S^{mr}v^n+S^{nr}v^m, \tag{A14}$$
$$S^{mn}=-S^{nm}, \quad \dot{S}^{mn}=0.$$

One can show that

$$M^{aibj:(r_N)}v_iv_j \quad \text{and} \quad M^{a^*ibj:(r_N)}v_iv_j$$

represent the vibrational and rotational multipole moment, respectively, of order 2^{N+2}.

A. *The Riemann Tensor*

Since

$$R^{ma}{}_{nb}=2(\gamma^{[m}{}_{[n}-\tfrac{1}{2}\delta^{[m}{}_{[n}\gamma)_{,}{}^{a]}{}_{b]},$$

with

$$\gamma^m{}_n=\eta_{nc}\gamma^{mc}, \quad \gamma=\eta_{mn}\gamma^{mn},$$

one finds with the help of (A11)

$$R^{manb}=\frac{{}_0N^{manb}}{r}+\frac{{}_0\mathrm{III}^{manb}}{r^2}+O(r^{-3}),$$

$${}_0N^{manb}=8k^{[a}k_{[b}\sum_{N=0}^{\infty}\frac{d^{N+4}}{du^{N+4}}M^{m]i}{}_{n]}{}^{j:(n_N)}k_ik_j(l_{r_N}),$$

$$\begin{aligned}{}_0\mathrm{III}^{manb}=8\sum_{Nm0}^{\infty}\Bigg\{(l_{r_N})\bigg[&k_ik_j(I_{[b}{}^{[a}+k^{[a}l_{[b}+k_{[b}l^{[a})\\&+2k_{(i}I_{j)[b}k^{[a}+2k_{(i}I_{j)}{}^{[a}k_{[b}\\&+(N+2)\bigg(\frac{(N+3)}{2}k_ik_j-2k_{(i}v_{j)}\bigg)k^{[a}k_{[b}\bigg]\\&+N(l_{r_{N-1}})k_ik_j[I_{r_N[b}k^{[a}+I_{r_N}{}^{[a}k_{[b}]\Bigg\}\\&\times\frac{d^{N+3}}{du^{N+3}}M^{m]i}{}_{n]}{}^{j:(r_N)}.\end{aligned} \tag{A15}$$

Hence, one has

$${}_0N^{manb}k_b={}_0N^{m^*anb}k_b=0,$$

$$\begin{aligned}{}_0\mathrm{III}^{manb}k_n&=-4k^b\sum_{N=0}^{\infty}(N+4)(l_{r_N})k^{[a}\frac{d^{N+3}}{du^{N+3}}M^{m]ijn:(r_N)}k_ik_jv_n,\\{}_0\mathrm{III}^{m^*anb}k_n&=4k^b\sum_{N=0}^{\infty}(N+4)(l_{r_N})k^{[a}\frac{d^{N+3}}{du^{N+3}}M^{m]ij^*n:(r_N)}k_ik_jv_n.\end{aligned} \tag{A16}$$

B. *The Superpotential*

From Eq. (1.3) one finds that the linear superpotential is

$$U_m{}^{[ns]}=2\eta_{ma}(\eta^{a[n}\gamma^{s]r}+\gamma^{a[n}\eta^{s]r}). \tag{A17}$$

Substituting (A13) into (A17) one has

$$\begin{aligned}U_m{}^{[ns]}=8\eta_{ma}\Bigg\{&\frac{1}{r}\sum_{N=0}^{\infty}(l_{r_N})k^{[s}\frac{d^{N+3}}{du^{N+3}}M^{n]iaj:(r_N)}k_ik_j\\&+\frac{1}{r^2}\bigg[mv^av^{[n}l^{s]}+\sum_{N=0}^{\infty}\bigg((l_{r_N})[2k_{(i}\delta_{j)}{}^{[s}\\&+(N+3)(k_ik_jl^s+\tfrac{1}{2}(N+2)k_ik_jk^{[s}\\&-2k_{(i}v_{j)}k^{[s})]+N(l_{r_{N-1}})k_ik_j\delta_{r_N}{}^{[s}\bigg)\\&\times\frac{d^{N+2}}{du^{N+2}}M^{n]iaj:(r_N)}\bigg]+O(r^{-3})\Bigg\}.\end{aligned}$$

Therefore, one obtains

$$U_m{}^{[ns]}k_n=\frac{4}{r^2}k^s\bigg[mv_m+\sum_{N=0}^{\infty}(l_{r_N})\frac{d^{N+2}}{du^{N+2}}\times M_m{}^{ijn:(r_N)}k_il_nv_j\bigg]. \tag{A18}$$